(a_k) convergent (Cauchy).

Let $e_n = \max\{a_k : 1 \le k \le n\}$

Prove or disprove (e_n) converges.

Let $a_n = \frac{1}{n}$

$e_1 = \max\{\frac{1}{1}\} = 1$, $e_2 = \max\{\frac{1}{1}, \frac{1}{2}\} = 1$, $e_3 = \max\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}\} = 1$

Let $a_n = -\frac{1}{n}$

$e_1 = \max\{-\frac{1}{1}\} = -1$, $e_2 = \max\{-\frac{1}{1}, -\frac{1}{2}\} = -\frac{1}{2}$

$a_3 = \max\{-\frac{1}{1}, -\frac{1}{2}, -\frac{1}{3}\} = -\frac{1}{3}, \dots$

$a_n = \frac{(-1)^n}{n} = \left(-\frac{1}{1}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \dots\right)$

$e_1 = \max\{-1\} = -1$, $e_2 = \max\{-1, \frac{1}{2}\} = \frac{1}{2}$,

$e_3 = \max\{-1, \frac{1}{2}, -\frac{1}{3}\} = \frac{1}{2}$, $e_4 = \max\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}\} = \frac{1}{2}$,

$e_5 = \max\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}\} = \frac{1}{2}, \dots$

$e_4 = \max\{a_1, a_2, a_3, a_4\}$ $\quad$ $e_5 = \max\{a_1, a_2, a_3, a_4, a_5\}$

$= \max\{e_4, a_5\}$

$e_4 \le e_5$

Last time:

Riemann rearrangement Thm.

If $\sum a_k$ is a series that converges but such that $\sum |a_k|$ diverges. [i.e. converges conditionally], then $\forall S \in \mathbb{R} \cup \{\pm\infty\}$, $\exists$ a rearrangement of the series that converges to S.

Example $\sum \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} \ldots$ ★

converge by AST.

but $\sum \left| \frac{(-1)^{k+1}}{k} \right| = \sum \frac{1}{k} \leftarrow$ harmonic, diverges.

$\therefore$ ★ converges conditionally.

Thus by RRT, given any number (eg 3), $\exists$ a rearrangement of ★ that converges to 3.

To do this, first add as many positive terms until the partial sum is >3. Then add negative terms until partial sum is <3. Then ... keep going.

Ratio Test.

Let $\sum a_k$ be a series, and suppose that $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists and $= R$.

Then (a) If $R < 1$, the series converges absolutely.

(b) If $R > 1$, the series diverges.

[If $R = 1$, there are some cases where the series diverges, other cases where the series converges.

eg: $\sum \frac{1}{n^p}$ always gives $R = 1$.]

<u>Pf.</u> Suppose the given conditions are met, and $R = \lim \left|\frac{a_{n+1}}{a_n}\right| < 1$.

Then choose r s.t. $R < r < 1$.

Then, letting $\varepsilon = r - R$, we see that $\exists\, N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$\left|\frac{a_{n+1}}{a_n}\right| - R \leq \left|\left|\frac{a_{n+1}}{a_n}\right| - R\right| < \overbrace{r - R}^{\varepsilon}$$

$$\Rightarrow \frac{|a_{n+1}|}{|a_n|} \leq r$$

$$\Rightarrow |a_{n+1}| \leq r|a_n| \quad \forall n \geq N.$$

let $n = N$

$|a_{n+3}| \leq r|a_{n+2}|$
$\leq r^2|a_{n+1}|$
$\leq r^3|a_n|$

$$\Rightarrow |a_{N+k}| \leq r^k |a_N| \quad \forall k \geq 0.$$

Then $\displaystyle\sum_{k=0}^{M} |a_{N+k}| \leq \sum_{k=0}^{M} r^k |a_N|$

geometric series that converges since $|r| < 1$.

$\therefore \displaystyle\sum_{k=0}^{\infty} |a_{N+k}|$ converges by comparison test.

$\therefore$ $\sum |a_n|$ converges by (eventually convergent)

$\Rightarrow$ $\sum a_n$ converges by abs convergence. $\boxed{//}$

For case $R > 1$,

choose $R > r > 1$.

use contrapositive of comparison test.

with geometric series, $r > 1$.